Laplace Transform for Linear Ordinary Differential Equation System.

$$\frac{dx_1(t)}{dt} = 3x_1(t) + 4x_2(t) + 6e^{2t}; \quad x_1(0) = 2$$

$$\frac{dx_2(t)}{dt} = 2x_1(t) + 5x_2(t) + 3t^2 + 2t; \quad x_2(0) = -3$$

$$\mathcal{L}\left\{\frac{dx_1}{dt}\right\} = \mathcal{L}\left\{3x_1 + 4x_2 + 6e^{2t}\right\}$$

$$\mathcal{L}\left\{\frac{dx_2}{dt}\right\} = \mathcal{L}\left\{2x_1 + 5x_2 + 3t^2 + 2t\right\}$$

$$s\mathcal{L}\{x_1\} - (2) = 3\mathcal{L}\{x_1\} + 4\mathcal{L}\{x_2\} + \frac{6}{s-2}$$

$$s\mathcal{L}\{x_2\} - (-3) = 2\mathcal{L}\{x_1\} + 5\mathcal{L}\{x_2\} + \frac{3\times2}{s^3} + \frac{2}{s^2}$$

$$\mathcal{L}\{x_1\} = \frac{1}{2}\left(-s\mathcal{L}\{x_2\} - 3 + 5\mathcal{L}\{x_2\} + \frac{6}{s^3} + \frac{2}{s^2}\right)$$

$$\mathcal{L}\{x_1\} = \left(\frac{-s}{2} + \frac{5}{2}\right)\mathcal{L}\{x_2\} + \frac{3}{s^3} + \frac{1}{s^2} - \frac{3}{2}$$

$$s\left(\left(-\frac{s}{2} + \frac{5}{2}\right)\mathcal{L}\{x_2\} + \frac{3}{s^3} + \frac{1}{s^2} - \frac{3}{2}\right) = +2 + 3\left(\left(-\frac{s}{2} + \frac{5}{2}\right)\mathcal{L}\{x_2\} + \frac{3}{s^3} + \frac{1}{s^2} - \frac{3}{2}\right) + 4\mathcal{L}\{x_2\} + \frac{6}{s-2}$$

$$\left(-\frac{s^2}{2} + \frac{5}{2}s\right)\mathcal{L}\{x_2\} + \frac{3}{s^2} + \frac{1}{s} - \frac{3}{2}s = \left(-\frac{3}{2}s + \frac{15}{2}\right)\mathcal{L}\{x_2\} + \frac{9}{s^3} + \frac{3}{s^2} - \frac{9}{2} + 2 + 4\mathcal{L}\{x_2\} + \frac{6}{s-2}$$

$$\left(-\frac{s^2}{2} + \frac{5s}{2}\right)\mathcal{L}\{x_2\} + \left(\frac{3}{2}s - \frac{15}{2} + 4\right)\mathcal{L}\{x_2\} =$$

$$= \cancel{-\frac{3}{s^2}} - \frac{1}{s} + \frac{3}{2}s + \frac{9}{s^3} \cancel{+\frac{3}{s^2}} - \frac{9}{2} + 2 + \frac{6}{s-2}$$

$$\left(s^2 + 4s - \frac{7}{2}\right)\mathcal{L}\{x_2\} = \frac{9}{s^3} - \frac{1}{s} - \frac{1}{2} + \frac{3}{2}s + \frac{6}{s-2}$$

$$= \frac{9(s-2) - s^2(s-2) - \frac{1}{2}s^3(s-2) + \frac{3}{2}s^4(s-2) + 6s^3}{(s-2)(s^3)}$$

$$\mathcal{L}\{x_2\} = \frac{9s - 18 - s^3 + 2s^2 - \frac{1}{2}s^4 + s^3 + \frac{3}{2}s^5 - 3s^4 + 6s^3}{(s-2)s^3\left(s^2 + 4s - \frac{7}{2}\right)}$$

$$\frac{d}{dt}\bar{x} = A\bar{x}$$

$$\mathcal{L}\left\{\frac{d}{dt}\bar{x}\right\} = \mathcal{L}\left\{A\bar{x}\right\}$$

$$s\mathcal{L}\{\bar{x}\} - \bar{x}(0) = A\mathcal{L}\{\bar{x}\}$$

$$s\mathcal{L}\{\bar{x}\} - A\mathcal{L}\{\bar{x}\} = \bar{x}(0)$$

$$(sI - A)\mathcal{L}\{\bar{x}\} = \bar{x}(0)$$

$$(sI-A)^{-1}(sI-A)\mathcal{L}\{\bar{x}\} = (sI-A)^{-1}\bar{x}(0)$$

$$\mathcal{L}\{\bar{x}\} = (sI-A)^{-1}\bar{x}(0)$$

$$\bar{x} = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}\bar{x}(0)$$

$$e^{A(t)} = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}$$

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}$$

$$(sI - A) = \begin{bmatrix} s-3 & -4 \\ -2 & s-5 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{|(sI-A)|} \begin{bmatrix} s-5 & 4 \\ 2 & s-3 \end{bmatrix}$$

$$= \frac{1}{(s-5)(s-3)-(-4)(-2)} \begin{bmatrix} s-5 & 4 \\ 2 & s-3 \end{bmatrix}$$

$$= \frac{1}{s^2-8s+15-8} \begin{bmatrix} s-5 & 4 \\ 2 & s-3 \end{bmatrix}$$

$$\mathcal{L}\{e^{At}\} = \begin{bmatrix} \frac{s-5}{s^2-8s+7} & \frac{4}{s^2-8s+7} \\ \frac{2}{s^2-8s+7} & \frac{s-3}{s^2-8s+7} \end{bmatrix}$$

$$\mathcal{L}\{e^{At}\} = \begin{bmatrix} \frac{A}{s-7} + \frac{B}{s-1} & \frac{C}{s-7} + \frac{D}{s-1} \\ \frac{E}{s-7} + \frac{F}{(s-1)} & \frac{G}{s-7} + \frac{H}{(s-1)} \end{bmatrix}$$

$$\frac{s-5}{s^2-8s+7} = \frac{A}{s-7} + \frac{B}{s-1}$$

$$s-5 = A(s-1) + B(s-7)$$

If $s=1$

$$(1)-5 = B(-6)$$

$$\frac{4}{6} = B \Rightarrow \frac{2}{3}$$

If $s=7$

$$(7)-5 = A(6)$$

$$\frac{2}{6} = A \Rightarrow \frac{1}{3}$$

$$\frac{s-3}{s^2-8s+7} = \frac{G}{s-7} + \frac{H}{(s-1)}$$

$$s-3 = G(s-1) + H(s-7)$$

If $s=1$

$$(1)-3 = H(-6)$$

$$H = \frac{1}{3}$$

If $s=7$

$$(7)-3 = G(6)$$

$$G = \frac{2}{3}.$$