Propiedades de la Transformada de Laplace

$$\begin{cases} \mathcal{L}\{f'(t)\} = sF(s) - f(0) \\ \mathcal{L}^{-1}\{F'(s)\} = -t f(t) \\ \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \dfrac{F(s)}{s} \\ \mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)\,d\sigma\right\} = \dfrac{f(t)}{t} \end{cases}$$

$t \in \mathbb{R}$

$s \in \mathbb{C}$

$f, F \in \mathbb{R}$

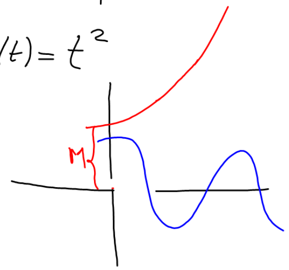
Teorema de Existencia de la Transformada de Laplace.

Dada $f(t)$ $t, f \in \mathbb{R}$

1.- $f(t)$ sea de orden exponencial

$$|f(t)| \leq M e^{At} \quad M, A \in \mathbb{R}$$

$$f(t) = t^2$$

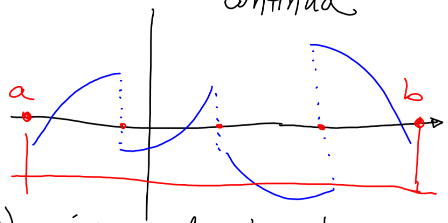


$$f(t) = e^{t^2}$$

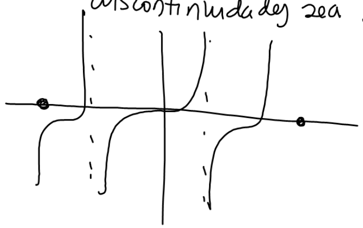
$$|e^{t^n}| \leq M e^{At}$$

$n > 1$

2°- $f(t)$ debe ser seccionalmente continua

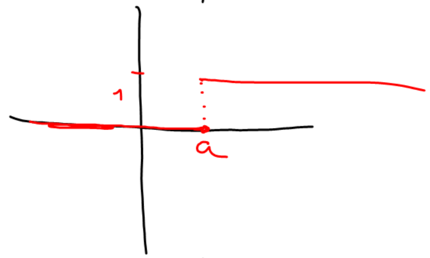


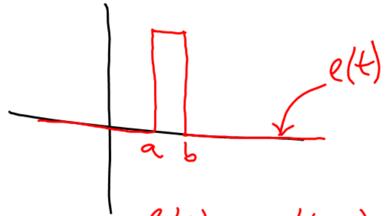
$f(t)$ será seccionalmente continua si en un rango cerrado $a \leq t \leq b$ el número de discontinuidades sea finito.



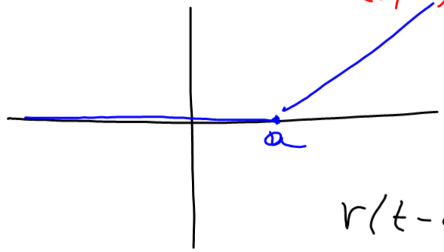
función escalón unitario

$$\mu(t-a) = \begin{cases} 0 & ; \ t<a \\ 1 & ; \ t>a \end{cases}$$



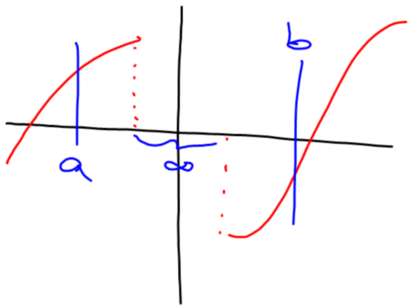


$$e(t) = \mu(t-a) - \mu(t-b).$$



$$r(t-a) = \begin{cases} 0 ; & t<a \\ t ; & a<t \end{cases}$$

$$\frac{d}{dt} r(t-a) = \mu(t-a)$$



Resumen Teorema

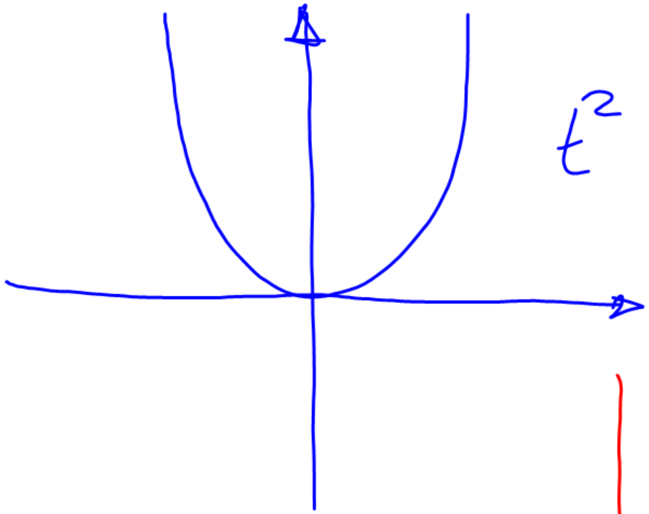
$f(t)$ $\quad t, f \in \mathbb{R}$

tendrá transformada de Laplace

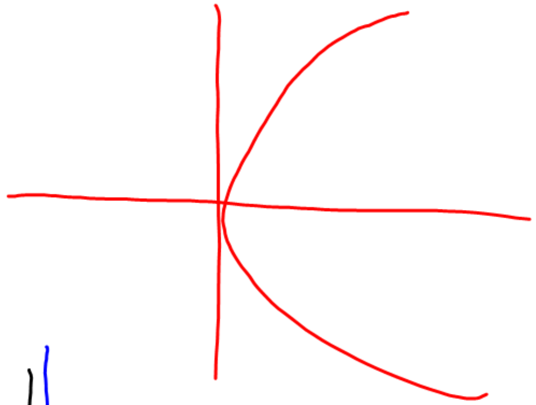
si

a) es de orden exponencial

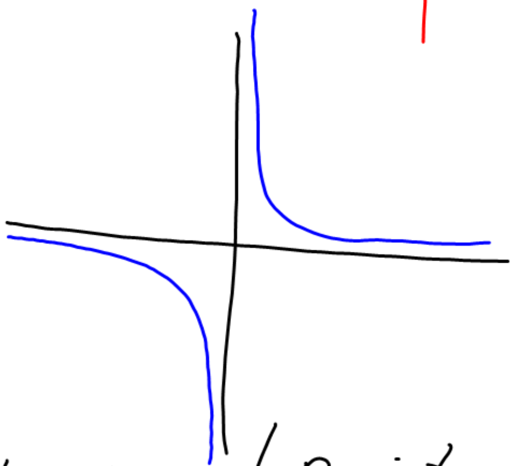
b) si es seccionalmente continua.



t^2



$\frac{1}{t}$



$|f(t)| \leq Me^{at}$

$$N(s,t) = \begin{cases} 0 & ; t < 0 \\ e^{-st} & ; t \geq 0 \end{cases}$$

Propiedades

1.- $\mathcal{L}\{af(t)+bg(t)\}=aF(s)+bG(s)$

2.- $\mathcal{L}\{f(at)\}=\frac{1}{a}F\left(\frac{s}{a}\right)$

3ab - deriv. e integrales

7.- $\mathcal{L}\{f(t-\tau)\}=e^{-s\tau}F(s)$

8.- $\mathcal{L}\{e^{at}f(t)\}=F(s-a)$

9.- $\mathcal{L}^{-1}\{F(s)\cdot G(s)\}=f(t)*g(t)$

Operador "convolución" $f(t)*g(t)=\int_0^t f(\tau)\cdot g(t-\tau)\,d\tau.$

EJEMPLO

$$f(t) = e^{3t}\cos(5t)$$

$$\mathcal{L}\{e^{3t}\} = \frac{1}{s-3} \qquad \mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{\cos(5t)\} = \frac{s}{s^2+25}$$

$$\mathcal{L}\{e^{3t}\cos(5t)\} = \frac{(s-3)}{(s-3)^2+25}$$

$$\mathcal{L}\{y(t)\} = \frac{8}{s^2+2s+2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{8}{s^2+2s+2}\right\}$$

$$= 8\mathcal{L}^{-1}\left\{\frac{1}{(s^2+2s+1)+1}\right\}$$

$$= 8\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+(1)^2}\right\}$$

$$= 8\cdot\left(e^{-t}\operatorname{sen}(t)\right)$$

$$\mathcal{L}\{\operatorname{sen} bt\} = \frac{b}{s^2+b^2} \qquad \mathcal{L}\{e^{at}\operatorname{sen}(bt)\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+3s+3}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{\left(s^2+3s+\frac{9}{4}\right)+3-\frac{9}{4}}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{\left(s+\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}$$

$$\mathcal{L}\{\operatorname{sen}(bt)\} = \frac{b}{s^2+b^2}$$

$$= \mathcal{L}^{-1}\left\{\frac{\left(s+\frac{3}{2}\right)-\frac{3}{2}}{\left(s+\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$\frac{\frac{3}{2}}{\frac{\sqrt{3}}{2}} = \frac{3}{\sqrt{3}} = \frac{3\sqrt{3}}{3} = \sqrt{3}$$

$$= \mathcal{L}^{-1}\left\{\frac{\left(s+\frac{3}{2}\right)}{\left(s+\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\} - \sqrt{3}\,\mathcal{L}^{-1}\left\{\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= e^{-\frac{3}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right) - \sqrt{3}\,e^{-\frac{3}{2}t}\operatorname{sen}\left(\frac{\sqrt{3}}{2}t\right).$$

EJEMPLO – 7 y 8 propiedad

$$L^{-1}\left\{\frac{e^{5s}}{(s+2)^3}\right\} = \left\{\begin{array}{l} 0 \;:\; t<5 \\ \frac{1}{2}e^{-2(t-5)}(t-5)^2 \end{array}\right\}$$

$$\frac{1}{2}L^{-1}\left\{\frac{1}{(s+2)^3}\right\} = \frac{1}{2}e^{-2t}t^2$$

$$\frac{1}{2}L^{-1}\left\{\frac{2}{s^3}\right\} = t^2$$

$$L^{-1}\left\{\frac{e^{5s}}{(s+2)^3}\right\} = \mu(t-5)\frac{e^{-2(t-5)}(t-5)^2}{2}$$

EJEMPLO - 9ª prop.

$$\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2s}{(s^2+4)^2}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\cdot\frac{2}{s^2+4}\right\}$$

$$= \frac{1}{2}\cos(2t) * \operatorname{sen}(2t).$$

$$\frac{1}{2}\cos(2t) * \operatorname{sen}(2t) = \frac{1}{2}\int_0^t \cos(2\tau)\cdot\operatorname{sen}(2(t-\tau))\,d\tau.$$

$$= \frac{1}{2}\int_0^t \cos(2\tau)\left(\operatorname{sen}(2t)\cdot\cos(2\tau) - \cos(2t)\cdot\operatorname{sen}(2\tau)\right)d\tau$$

$$= \frac{\operatorname{sen}(2t)}{2}\int_0^t \cos^2(2\tau)\,d\tau - \frac{\cos(2t)}{2}\int_0^t \operatorname{sen}(2\tau)\cos(2\tau)\,d\tau$$

$$= \frac{\operatorname{sen}(2t)}{2}\int_0^t \left(\frac{1}{2}+\frac{1}{2}\cos(4\tau)\right)d\tau - \frac{\cos(2t)}{4}\int_0^t \operatorname{sen}(2\tau)\left(2\cos(2\tau)\,d\tau\right)$$

$$= \frac{\operatorname{sen}(2t)}{4}\int_0^t d\tau + \frac{\operatorname{sen}(2t)}{16}\int_0^t \cos(4\tau)\,4d\tau - \frac{\cos(2t)}{8}\left(\operatorname{sen}(2\tau)\right)^2\Big]_0^t$$

$$= \frac{\operatorname{sen}(2t)}{4}\,\tau\Big|_0^t + \frac{\operatorname{sen}(2t)}{16}\cdot\operatorname{sen}(4\tau)\Big]_0^t - \frac{\cos(2t)}{8}\operatorname{sen}^2(2\tau)\Big]_0^t$$

$$= \frac{t\operatorname{sen}(2t)}{4} + \frac{\operatorname{sen}(2t)\cdot\operatorname{sen}(4t)}{16} - \frac{\cos(2t)}{8}\operatorname{sen}^2(2t)$$