Laplace Transform

$$\mathcal{L}\{f(t)\} = F(s)$$

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$ warning: "is not unique"

$s \in \mathbb{C}$ $F \in \mathbb{R}$ $t, f \in \mathbb{R}$

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{st} F(s)\, ds$$

$\mathcal{L}\{f(t)\} = F(s) \qquad \mathcal{L}^{-1}\{F(s)\} = f(t)$

① $\mathcal{L}\{af(t)+bg(t)\} = aF(s) + bG(s) \quad a,b\in\mathbb{R}$

② $a>0 \quad \mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$

$f(t) = 5t^2 - 8e^{3t} + 4\operatorname{sen}(t)$

$\mathcal{L}\{f(t)\} = 5\mathcal{L}\{t^2\} - 8\mathcal{L}\{e^{3t}\} + 4\mathcal{L}\{\operatorname{sen}(t)\}$

$= 5\left(\frac{2!}{s^3}\right) - 8\left(\frac{1}{s-3}\right) + 4\left(\frac{1}{s^2+1}\right)$

$= \frac{10}{s^3} - \frac{8}{s-3} + \frac{4}{s^2+1}$

$\mathcal{L}\{f(t)\} = F(s) \qquad \mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$

$\mathcal{L}\{e^t\} = \frac{1}{s-1} \qquad \mathcal{L}\{e^{2t}\} = \frac{1}{2}\left(\frac{1}{\left(\frac{s}{2}\right)-1}\right)$

$= \frac{1}{2}\left(\frac{1}{\frac{s-2}{2}}\right)$

$\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad = \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{s-2}\right)$

$\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}$

$\mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1}$

$\mathcal{L}\{\cos(4t)\} = \frac{1}{4}\left(\frac{\left(\frac{s}{4}\right)}{\left(\frac{s}{4}\right)^2+1}\right)$

$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2} \qquad = \frac{1}{16}\left(\frac{s}{\left(\frac{s^2+16}{16}\right)}\right)$

$= \frac{1}{\cancel{16}}\left(\frac{\cancel{16}s}{s^2+16}\right)$

$\mathcal{L}\{\cos(t)\} = \frac{s}{s^2+b^2}$

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$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

GP $$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$$

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$$

$$\mathcal{L}\{f'''(t)\} = s^3 F(s) - s^2 f(0) - s f'(0) - f''(0)$$

growth.

ODEL (2) cc H.

$$\begin{cases} x(0) = 4 \\ x'(0) = -8 \end{cases}$$

$$\frac{d^2 x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 6x(t) = 0$$

$$\mathcal{L}\left\{\frac{d^2x}{dt^2}\right\} - 5\mathcal{L}\left\{\frac{dx}{dt}\right\} + 6\mathcal{L}\{x(t)\} = 0\mathcal{L}\{1\}$$

$$= 0$$

$$\left[s^2X(s) - s\cdot(4) - (-8)\right] - 5\left[sX(s) - (4)\right] + 6X(s) = 0$$

$$\left(s^2 - 5s + 6\right)X(s) - 4s + 8 + 20 = 0$$

$$\left(s^2 - 5s + 6\right)X(s) = 4s - 28 \qquad X \in \mathbb{R} \quad s \in \mathbb{C}.$$

$$\boxed{X(s) = \frac{4s - 28}{s^2 - 5s + 6}}$$

$\mathcal{L}$TPS

$$X(s) = \frac{4s-28}{s^2-5s+6}$$

$$\frac{4s-28}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3}$$

Partial Fraction Method.

$$4s-28 = A(s-3) + B(s-2)$$

if $s=3$

$$4(3)-28 = A(0) + B(3-2)$$

$$-16 = B \Rightarrow \boxed{B=-16}$$

if $s=2$

$$4(2)-28 = A(2-3) + (-16)(0)$$

$$-20 = A(-1) \Rightarrow \boxed{A=20}$$

$$X(s) = \frac{20}{s-2} - \frac{16}{s-3}$$

$$\boxed{\begin{aligned} \mathcal{L}\{e^{at}\} &= \frac{1}{s-a} \\ \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} &= e^{at} \end{aligned}}$$

$$X(s) = 20\left(\frac{1}{s-2}\right) - 16\left(\frac{1}{s-3}\right)$$

$$x(t) \Rightarrow \mathcal{L}^{-1}\{X(s)\} = 20\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 16\,\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$\boxed{x(t) = 20e^{2t} - 16e^{3t}}$$

Particular Solution.

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$$\mathcal{L}^{-1}\{F'(s)\} = -t f(t)$$

$$\mathcal{L}^{-1}\{F^{(n)}(s)\} = (-1)^n t^n f(t)$$

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$$\mathcal{L}\left\{\int_0^t f(z)\,dz\right\} = \frac{F(s)}{s}$$

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$$\mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)\,d\sigma\right\} = \frac{f(t)}{t}$$

⑦ $\mathcal{L}\{f(t-\tau)\} = e^{-s\tau}F(s)$

⑧ $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

$$\mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}$$

$$\mathcal{L}\{e^{at}\cos(bt)\} = \frac{(s-a)}{(s-a)^2+b^2}$$

$\mathcal{L}\{1\} = \frac{1}{s}$ $\quad \mathcal{L}\{e^{at}\cdot(1)\} = \frac{1}{s-a}$

$\mathcal{L}\{t\} = \frac{1}{s^2}$ $\quad \mathcal{L}\{e^{at}\cdot(t)\} = \frac{1}{(s-a)^2}$

$\mathcal{L}\{t^2\} = \frac{2!}{s^3}$ $\quad \mathcal{L}\{e^{at}t^2\} = \frac{2!}{(s-a)^3}$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+2s+1)+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2+(1)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s+1)-1}{(s+1)^2+(1)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2+(1)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+(1)^2}\right\}$$

$$= e^{-t}\cos(t) - e^{-t}\operatorname{sen}(t)$$