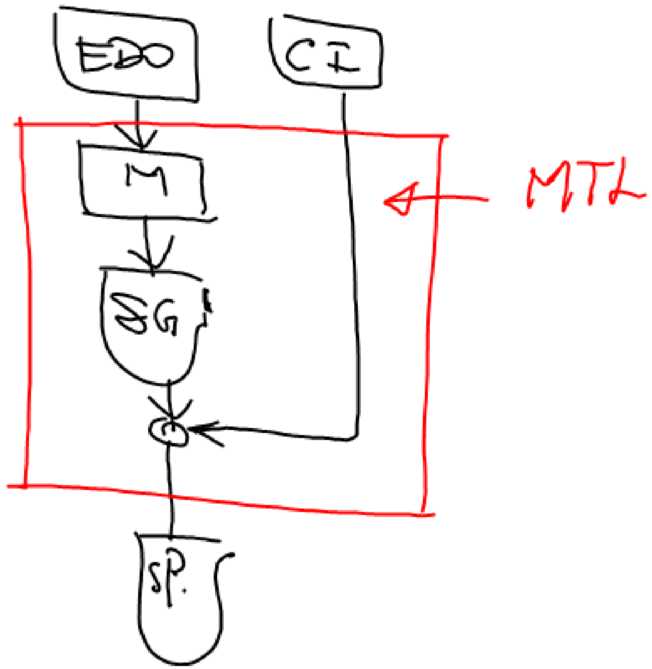
Capítulo IV: La Transformada de Laplace.

"Es método para resolver problemas con Ecuaciones Diferenciales de condiciones iniciales"

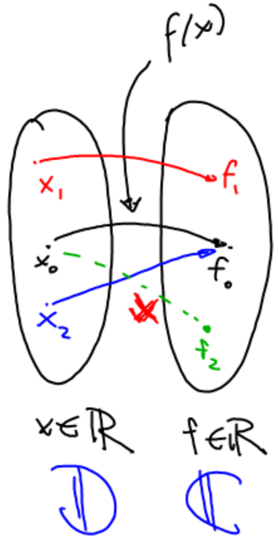


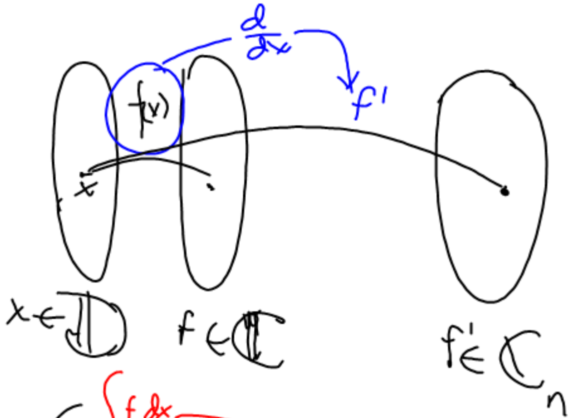
La Transformada

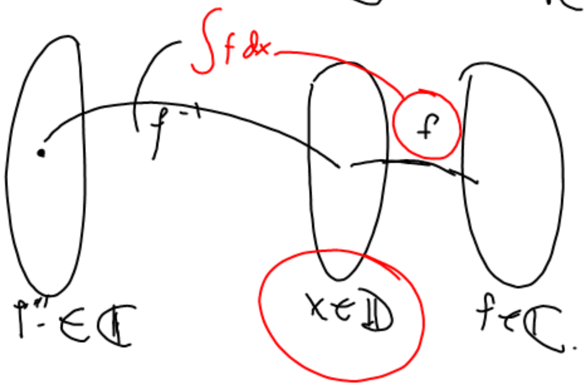
x	x^2
2	4
−2	4

x	$\sqrt{x}$ X
4	2
~~4~~	~~2~~

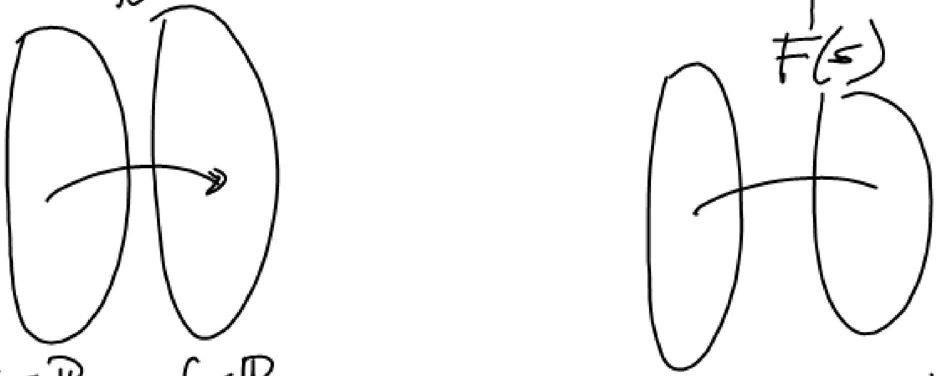
x	x^2	$2x$	$\frac{x^3}{3}$
3	9	6	9
−3	9	−6	−9







$f(x) \longrightarrow T\{f\} \longrightarrow F(s)$



$x \in \mathbb{R}$ $\quad f \in \mathbb{R}$ $\qquad s \in \mathbb{C}$ $\quad F \in \mathbb{R}$

complejo

Lineal

$a f(x) \quad a \in \mathbb{R} \longrightarrow a F(s)$

$f(x) + g(x) \longrightarrow F(s) + G(s)$

$\frac{d}{dx} f(x) \longrightarrow s F(s)$

$\frac{d^2}{dx^2} f(x) \longrightarrow s^2 F(s)$

$\int f \, dx \longleftarrow \frac{F(s)}{s}$

$$T\{f(t)\} = \int_{-\infty}^{\infty} N(t,s)\, f(t)\, dt$$

operador

argumento

núcleo

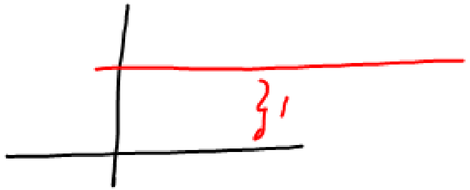
Transformada de Laplace

$$N(t,s) = \begin{cases} 0 & t<0 \\ e^{-st} & t \geq 0 \end{cases}$$

$\mathcal{L}$

$$\boxed{\mathcal{L}\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t)\, dt}$$

$f(x) = 1$



$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\cdot(1)\cdot dt$$

$$= \left[\int e^{-st}\,dt\right]_0^\infty$$

$$= \left(\frac{e^{-st}}{-s}\right)_0^\infty$$

$$= -\frac{1}{s}\left(e^{-st}\right)_0^\infty$$

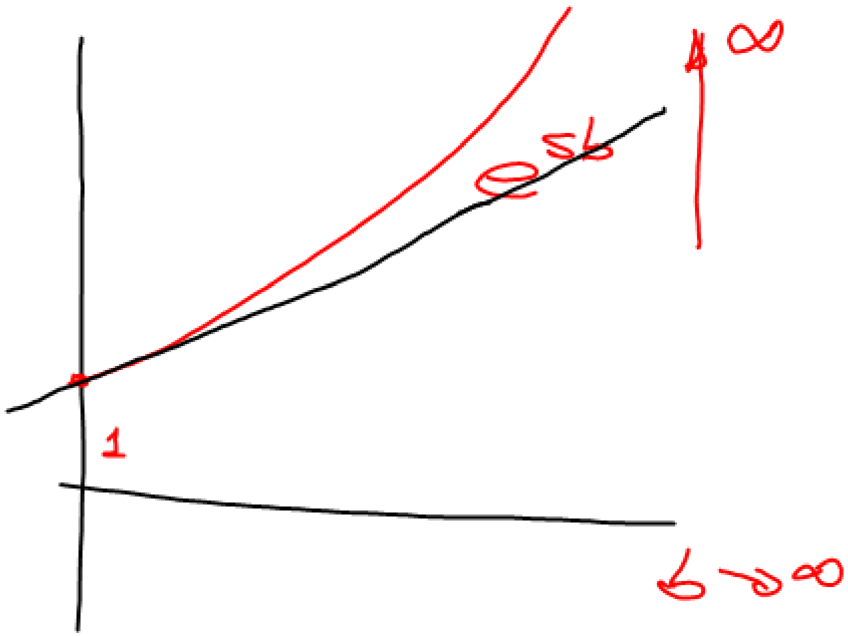
$$= -\frac{1}{s}\left(\cancelto{0}{\lim_{b\to\infty} e^{-sb}} - 1\right)$$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}}$$

$s \in$ Compleja

$x \in \mathbb{R}$

$$\lim_{b\to\infty} e^{-sb} = \lim_{b\to\infty} \frac{1}{e^{sb}} \Rightarrow \lim_{a\to\infty} \frac{1}{a} = 0$$



$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

$t \in \mathbb{R}^+$

$$= \left[\int t e^{-st} dt \right]_0^\infty$$

$$-\frac{(1+st)\,e^{-st}}{s^2}$$

$$= \left[-\frac{e^{-st}}{s^2} - \frac{t e^{-st}}{s} \right]_0^\infty$$

$$= -\frac{1}{s^2}\left(e^{-st}\right)_0^\infty - \frac{1}{s}\left(t e^{-st}\right)_0^\infty$$

$$= -\frac{1}{s^2}\left(\lim_{b\to\infty} e^{-sb} - 1\right) - \frac{1}{s}\left(\lim_{b\to\infty} b \cdot \lim_{b\to\infty} e^{-sb} - 0\right)$$

$$= -\frac{1}{s^2}\left((0) - 1\right) - \frac{1}{s}\left((0) - (0)\right)$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

$f(t)$	$\mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n, $n\in\mathbb{N}$	$\frac{n!}{s^{n+1}}$
e^{at}, $a\in\mathbb{R}$	$\frac{1}{s-a}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\operatorname{sen}(bt)$, $b\in\mathbb{R}$	$\frac{b}{s^2+b^2}$

$$\mathcal{L}\{\cos(3t)\} = \frac{s}{s^2+9}$$

$$\mathcal{L}\{\operatorname{sen}(2t)\} = \frac{2}{s^2+4}$$

$$\mathcal{L}\{t^6\} = \frac{6!}{s^7}$$

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \left(\int e^{-st} f'(t)\,dt\right)_0^\infty$$

$u = e^{-st}$
$du = -se^{-st}dt$
$dv = f'dt$
$v = f$

$$= \left(fe^{-st} - \int -sfe^{-st}dt\right)_0^\infty$$

$$= \left(fe^{-st}\right)_0^\infty + s\int_0^\infty e^{-st} f\,dt$$

$$= \left(\lim_{t\to\infty} e^{-st} - f(0)\right) + s\mathcal{L}\{f\}.$$

$$\boxed{\mathcal{L}\{f'(t)\} = sF(s) - f(0)}$$

$$\mathcal{L}\{f''\} = s^2F(s) - s\cdot f(0) - f'(0)$$

$$\mathcal{L}\{f'''\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

$$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = e^t \qquad y(0) = 4 \quad y'(0) = -3$$

$$\mathcal{L}\left\{\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y\right\} = \mathcal{L}\left\{e^t\right\}$$

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + \mathcal{L}\left\{-5\frac{dy}{dt}\right\} + \mathcal{L}\left\{6y\right\} = \frac{1}{s-1}$$

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} - 5\mathcal{L}\left\{\frac{dy}{dt}\right\} + 6\mathcal{L}\{y\} = \frac{1}{s-1}$$

$$\left(s^2\mathcal{L}\{y\} - s\cdot y(0) - y'(0)\right\} - 5\left(s\mathcal{L}\{y\} - y(0)\right) + 6\mathcal{L}\{y\} = \frac{1}{s-1}$$

$$s^2\mathcal{L}\{y\} - 4s + 3 - 5s\mathcal{L}\{y\} + 20 + 6\mathcal{L}\{y\} = \frac{1}{s-1}$$

$$\boxed{(s^2 - 5s + 6)\mathcal{L}\{y\} - 4s + 23 = \frac{1}{s-1}}$$

$$\boxed{\mathcal{L}\{y\} = \frac{\frac{1}{s-1} + 4s - 23}{(s^2 - 5s + 6)}}$$